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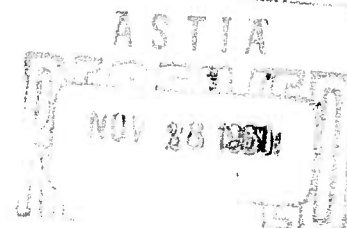
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**OPTIMUM INTERCEPTION
OF A BALLISTIC MISSILE
AT MODERATE RANGE**

**BY
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**RESEARCH PAPER No. 25
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AT MODERATE RANGE**

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OPTIMUM INTERCEPTION OF A BALLISTIC MISSILE AT MODERATE RANGE

The problem of intercepting a ballistic missile optimally at moderate range is discussed in this report. The problems of minimum time and minimum fuel consumption are discussed specifically. A procedure is given for determining the trajectory on a digital computer, and some necessary and sufficient conditions are given for the corresponding optima.

The computational procedure is a method of successive approximations. The simpler cases have been checked out on the digital computer and further programming is under way. In the examples run, each iteration takes somewhat less than one second computing time on the CDC 1604. From three to six iterations are usually required to determine a trajectory, so that the trajectory is determined in five seconds or so.

The variational equations which are used to determine the trajectory can also be used to determine the corrective thrust should the actual trajectory deviate slightly from the planned trajectory.

The following simplifying assumptions are made. The speed c' of the emitted gases of the rocket is assumed to be a constant of the rocket. The action takes place above the sensible atmosphere so that aerodynamic forces are negligible. The particular cases are considered wherein the range of the action is small enough that gravity is constant during the flight or can be approximated as a linear function of displacement; the case of central body motion is also discussed very briefly. Much of the paper is restricted to the case of plane motion, where the motion of the rocket and the target lie in a properly defined plane. This is for simplicity in programming and not a restriction on the method. Most of the procedure is elementary and hence the paper is somewhat expository.

2. Basic equations

The equation of motion of a rocket in a gravitational field and with no outside forces may be written as

$$(1) \quad \ddot{\vec{r}} = \vec{g} + \vec{a},$$

where \vec{r} is the position vector, \vec{g} is the acceleration due to gravity, \vec{a} is the acceleration due to thrust, and a dot (·) over a variable indicates its time derivative. We may write

$$(2) \quad \vec{a} = -c'\dot{m}/(1-m)\vec{e},$$

where m is the ratio of the mass of fuel which has been consumed to the initial gross mass of the rocket and \vec{e} is a unit vector in the direction of the thrust.

A useful kinematic relation is the following,

$$(3) \quad \int_0^t a \, dt = c' \int_0^t \frac{\dot{m} \, dt}{1-m} = -c' \ln(1-m).$$

Since the fuel consumed is proportional to m and $\ln(1-m)$ is a monotonic function of m , conditions involving the final value of mass may be rephrased in terms of the integral of the acceleration, subject to the constraints on the size of \dot{m} . For practical purposes we may take \dot{m} as being bounded above by a constant, and bounded below by zero, and it may be chosen anywhere between or on these bounds as long as any fuel remains.

Integrated form of the equations of motion. For the case where gravity may be considered constant, the equations of motion may be written

$$(4) \quad \begin{cases} \ddot{x} = a \cos p \\ \ddot{y} = a \sin p - g \end{cases}$$

for two-dimensional motion, with the y axis vertical. If gravity may be satisfactorily approximated by a linear function of the displacement, these become

$$(5) \quad \begin{cases} \ddot{x} = -b^2 x + a \cos p, \\ \ddot{y} = +B^2 y + a \sin p - g, \end{cases}$$

where g is the gravitational acceleration at the origin of coordinates, $b^2 = g/r_E$, with r_E the distance from the

origin of coordinate to the earth's center and $B^2 = 2b^2$.

The corresponding displacement may be expressed at any time T as

$$(6) \quad \begin{cases} x(T) = x_0 + \dot{x}_0 T + \int_0^T (T-t) a \cos p \, dt \\ y(t) = y_0 + \dot{y}_0 T - gT^2/2 + \int_0^T (T-t) a \sin p \, dt \end{cases}$$

and

$$(7) \quad \begin{cases} x(T) = x_0 \cos bT + \frac{\dot{x}_0}{b} \sin bT + \frac{1}{b} \int_0^T a \sin b(T-t) \cos p \, dt \\ y(T) = (y_0 + \frac{g}{B^2}) \cosh BT + (\dot{y}_0/B) \sinh BT - g/B^2 \\ \quad + (1/B) \int_0^T a \sinh B(T-t) \sin p \, dt. \end{cases}$$

3. Extremals.

The curves which correspond to optimum motion from one point to another are called extremals. They may be characterized for the case where the velocity does not enter in the terminal conditions (and if gravity is linear, for $T < \pi/2b$) as follows.

First, the thrust must be a maximum for a properly chosen initial period, and zero thereafter. This is a consequence of the fact that the coefficient of $a \cos p$ and $a \sin p$ in equations (6) and (7) diminish with time.

Second, the direction of the thrust is given by

$$(8) \quad \tan p = c,$$

or

$$(9) \quad \tan p = c \frac{(1/B) \sinh B(T-t)}{(1/b) \sin b(T-t)},$$

accordingly as the equations of motion are taken as (4) or (5). Equation (8) or (9) is called the steering equation.

An elementary proof is given in section (5) to show that these conditions are sufficient to describe an extremal, and henceforth only trajectories wherein the thrust satisfies these conditions will be considered.

The equations for the motion of a rocket at any time (6) and (7) will be replaced by

$$(10) \quad \begin{cases} x(T) = x_0 + \dot{x}_0 T + \int_0^{t_1} (T-t) a \cos p \, dt \\ y(T) = y_0 + \dot{y}_0 T - gT^2/2 + \int_0^{t_1} (T-t) a \sin p \, dt, \end{cases}$$

where $\tan p = c$, and

$$(11) \quad \begin{cases} x(T) = x_0 \cos bT + (\dot{x}_0/b) \sin bT + \\ \quad + (1/b^2) \int_0^{t_1} (a/R) \sin^2 b(T-t) \, dt, \\ y(T) = (y_0 + g/B^2) \cosh BT + (\dot{y}_0/B) \sinh BT - g/B^2 \\ \quad + (c/B^2) \int_0^{t_1} (a/R) \sinh^2 B(T-t) \, dt, \end{cases}$$

where $R = \sqrt{(1/b)^2 \sin^2 b(T-t) + (c/B)^2 \sinh^2 B(T-t)}$, and $t_1 \leq T$. The problem of determining an optimum trajectory is then reduced to that of finding c , t_1 , and T , since these characterize an extremal.

4. Differentials of extremals.

Formulas are derived here for the changes in terminal values of the coordinates due to small changes in the parameters of the extremals.

Let us consider now two extremals, near together in the following sense. One extremal corresponds to a set of values p, t_1, T , and the other corresponds to a neighboring set $p+\delta p, t_1+\delta t_1, T+\delta T$, in equations (10). Then the resulting change in the terminal values of x and y are

$$(12) \quad \begin{aligned} \delta X &= (\dot{x}_0 + \int_0^{t_1} a \cos p \, dt) \delta T + [(T-t) a \cos p]_{t_1} \delta t_1 \\ &\quad - \int_0^{t_1} a \sin p \, dt \, \delta p, \\ \delta Y &= (\dot{y}_0 + \int_0^{t_1} a \sin p \, dt) \delta T + [(T-t) a \sin p]_{t_1} \delta t_1 + \end{aligned}$$

$$+ \int_0^{t_1} (T-t) a \cos p \, dt \, \delta p,$$

for gravity constant if all initial values are specified.

If the linear approximation to the gravitational field is used, the equations for the variations are more involved. In the following equations let C, S, Ch, Sh denote $\cos b(T-t)$, $\sin b(T-t)$, $\cosh B(T-t)$, $\sinh B(T-t)$, respectively. Then

$$R = \sqrt{(S/b)^2 + (c \, Sh/B)^2}, \text{ and}$$

$$(13) \left\{ \begin{aligned} \delta X &= \left[\int_0^{t_1} a (\cos^3 p \, C + 2 \sin^2 p \cos p \, C - c \cos^2 p \sin p \, Ch) dt \right. \\ &\quad \left. - bx_0 \sin bT + \dot{x}_0 \cos bT \right] \delta T + (aS^2/Rb^2)_{t_1} \delta t_1 \\ &\quad - (1/c) \int_0^{t_1} a R \sin^2 p \cos^2 p \, dt \, \delta c \\ \delta Y &= \left[\int_0^{t_1} a (2 \cos^2 p \sin p \, Ch + \sin^3 p \, Ch - \frac{1}{c} \sin^2 p \cos p \, C) dt \right. \\ &\quad \left. + (By_0 + \frac{g}{B}) \sinh BT + \dot{y}_0 \sinh BT \right] \delta T + (ac \, Sh^2/RB^2)_{t_1} \delta t_1 \\ &\quad + (1/c^2) \int_0^{t_1} a R \sin^2 p \cos^2 p \, dt \, \delta c, \end{aligned} \right.$$

$$\text{where } \tan p = \frac{cb}{B} \frac{Sh}{S}.$$

5. Fixed point in fixed time, with minimum fuel consumption.

Let us now consider the problem of sending the rocket to the prescribed point X_2, Y_2 in specified time T , with minimum fuel consumption. The initial values are all assumed to be given. The problem is that of finding the values of c, t_1 so that $x(T) = X_2$, $y(T) = Y_2$.

Suppose we guess a pair of values for t_1 and c (or p), and compute the corresponding trajectory. The end values will not be correct generally; let the errors be

$$(14) \quad \begin{cases} E = X - X_2 \\ F = Y - Y_2 \end{cases}$$

Now set

$$(15) \quad \begin{cases} \delta X = -E \\ \delta Y = -F \end{cases}$$

and use the formulas developed for $\delta X, \delta Y$. Then

$$(16) \quad \begin{cases} [(T-t)a \cos p]_{t_1} \delta t_1 - \int_0^{t_1} (T-t) a \sin p \, dt \, \delta p = -E \\ [(T-t)a \sin p]_{t_1} \delta t_1 + \int_0^{t_1} (T-t) a \cos p \, dt \, \delta p = -F \end{cases}$$

for constant gravity, and for the linearized equations

$$(17) \quad \begin{cases} (a s^2 / R b^2)_{t_1} \delta t_1 - \frac{1}{c} \int_0^{t_1} a R \sin^2 p \cos^2 p \, dt \, \delta c = -E \\ (a s^2 / R b^2)_{t_1} \delta t_1 + \frac{1}{c^2} \int_0^{t_1} a R \sin^2 p \cos^2 p \, dt \, \delta c = -F \end{cases}$$

The values of δp (or δc), δt_1 determined from this yield new values for p, t_1 . Using these, we compute a new trajectory and correct, repeating until $E^2 + F^2$ is less than some preassigned value, a convergence criterion. No convergence problems were encountered and a brief discussion of the results is given in section 9.

6. Proof of minimum fuel consumption. A Mayer reciprocal relation.

The proof is given here that the curve C^* found in the previous section has the lowest value for $\int_0^T a \, dt$ for admissible paths. An admissible path is one whereon the bounds on a are satisfied and the proper initial and final values are assumed. A corollary result is the equivalence between problems of specified displacement with minimum fuel consumption and those

of specified fuel consumption and maximum displacement, etc. known as Mayer reciprocal relations.

The proof will be carried through for equations (5); the proof for constant gravity is given in other reports already. Suppose we have found c, t_1 in the previous section so that E, F are zero. Denote by an asterisk (*) the quantities on that path C^* and by capital letters the corresponding quantities on any other admissible path.

If in equations (7) the second is multiplied by c and the two are added, the resulting equation may be rewritten, for any admissible path

$$(18) \quad (X_2 - x_0 \cos bT - \frac{\dot{x}_0}{b} \sin bT) + c[Y_2 - (y_0 + \frac{E}{B^2}) \cosh BT - \frac{\dot{y}_0}{B} \sinh BT + \frac{E}{B^2}] \\ = \int_0^T a[\frac{1}{b} \cos p + \frac{c}{B} \sin p] dt .$$

Now everything on the left side of this equation is completely determined, either from the initial conditions and final conditions, or from c which has been found. Hence the right side is also determined.

Now suppose there is another admissible path for which the fuel consumption does not exceed that on C^* . Then

$$(19) \quad \int_0^T A dt \leq \int_0^T a^* dt = \int_0^T a^* dt .$$

Let

$$(20) \quad \int_0^{t_1} (a^* - A) dt = D .$$

Then, from (19) and (20)

$$(21) \quad \int_{t_1}^T A dt \leq D .$$

Now subtract the expressions obtained from (18) for the two paths.

$$(22) \quad \int_0^{t_1} a^* \left(\frac{1}{b} S \cos p^* + \frac{c}{B} Sh \sin p^* \right) dt \\ - \int_0^T A \left(\frac{1}{b} S \cos P + \frac{c}{B} Sh \sin P \right) dt = 0$$

Now, from equation (9)

$$(23) \quad \frac{1}{b} S \cos p^* + \frac{c}{B} Sh \sin p^* = R,$$

where

$$R^2 = \left(\frac{1}{b} \right)^2 S^2 + \left(\frac{c}{B} \right)^2 Sh^2$$

as defined earlier. Note that R is a decreasing function of t for $bT < \pi/2$, and $R(T) = 0$. In view of this (22) may be rewritten

$$(24) \quad \int_0^{t_1} (a^* - A) R dt + \int_0^{t_1} A R [1 - \cos(p^* - P)] dt - \int_{t_1}^T A R \cos(p^* - P) dt \\ = 0.$$

Now

$$\int_0^{t_1} (a^* - A) R dt \geq R(t_1) \int_0^{t_1} (a^* - A) dt \\ = r(t_1) D$$

$$\int_{t_1}^T A R \cos(p^* - P) dt \leq \int_{t_1}^T A R dt \\ \leq R(t_1) \int_{t_1}^T A dt \\ \leq D R(t_1),$$

so that the sum of the first and last integrals in (24) is positive or zero. The second integral is also clearly positive or zero. Inspection reveals further that the integrals in (24) have a sum which is not zero unless $a^* = A$ except on a set of measure zero. Hence the curve found in the preceding section leads to a minimum value for the fuel consumption for that value of T and is unique.

Mayer-reciprocal relation. Actually the trajectory C^* just found represents an optimum relation among the set of variables $x_0, y_0, \dot{x}_0, \dot{y}_0, X, Y, m_2$. For consider all by one of these as being prescribed as on C^* . For example, take all but x_0 as having prescribed values. Let an admissible curve be one on which $y_0, \dot{x}_0, \dot{y}_0, X, Y, m_2$ assume the values they assume on C^* , with T fixed.

Then x_0 assumes its minimum value on C^* for admissible curves.

To show this, let us consider equation (18) for the two curves. Everything on the left is prescribed except the term involving x_0 . Hence

$$(25) \quad (-x_0^* + X_0) \cos bT = \int_0^{t_1} a^* R \, dt - \int_0^T AR \cos(p^* - P) \, dt$$

Now the right side of this equation is positive unless $\bar{A} = \bar{a}^*$, by the arguments given earlier in this section. Since $0 < bT < \pi/2$, $x_0^* < X_0$ unless $\bar{A} = \bar{a}^*$.

Similarly, if c is positive, y_0 is a minimum if all the other quantities are fixed as on C^* . We see that if all but one of the set is fixed and if c is positive, then C^* defines a minimum for $x_0, y_0, \dot{x}_0, \dot{y}_0, m_2$ and a maximum for x_2, y_2 .

This is the Mayer-reciprocal relation among the set just discussed on C^* . In some cases, T may be included in the set but generally it may not be. The important concept in the Mayer-reciprocal relation is that an extremal furnishes not just a maximum or a minimum for one variable but rather an optimum relation among a set of variables. Throughout the problems solved in this paper, we seek a curve such as C^* which satisfies just the specified conditions.

8. Interception in minimum time for incoming target.

The problem of intercepting a ballistic missile in minimum time is solved here. The equations used are those for the case where gravity is constant.

The relative motion of the missile and the target may be written, in a suitably chosen coordinate set as

$$(25) \begin{cases} E = X_0 + \dot{X}_0 t - \int_0^{t_1^*} (t-\tau) a \cos p \, d\tau, \\ F = Y_0 + \dot{Y}_0 t - \int_0^{t_1^*} (t-\tau) a \sin p \, d\tau, \\ G = Z_0 + \dot{Z}_0 t = 0. \end{cases}$$

The number t_1^* is determined as follows. There is a maximum value $t_{1\max}$ for t_1 , and in equations (25), $t_1^* = t$ if $t < t_{1\max}$ and $t_1^* = t_{1\max}$ if $t > t_{1\max}$. Part of the computing problem is to make this decision.

To start the computing routine, we chose the first estimate of p so that the direction was about 60° from the initial line of sight from rocket to target. The equations of motion were then integrated until $E^2 + F^2$ began to increase. This determined the first estimate of T . Then the variational equations (12) were used with (25)

$$(26) \begin{cases} \delta E_2 = (\dot{X}_0 - \int_0^{t_1^*} a \cos p \, dt) \delta T + \int_0^{t_1^*} (T-t) a \sin p \, dt \, \delta p \\ \delta F_2 = (\dot{Y}_0 - \int_0^{t_1^*} a \sin p \, dt) \delta T - \int_0^{t_1^*} (T-t) a \cos p \, dt \, \delta p. \end{cases}$$

To get new estimates of T, p set

$$(27) \begin{cases} \delta E_2 = -E_2 \\ \delta F_2 = -F_2 \end{cases}$$

and solve for $\delta T, \delta p$. This defines a new approximating trajectory. For this and subsequent iterations, the value of T is determined in this fashion, rather than by seeking a minimum value for $E^2 + F^2$. The computation is terminated when $E_2^2 + F_2^2$ is below a prescribed number.

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The trajectory obtained above could represent a maximum or a minimum value of T ; there are various ways to check. One way is to diminish t_1 by an amount $-\delta t_1$ and calculate the corresponding increment δT . If $\delta T/\delta t_1 < 0$, then T is at least a relative minimum. The case where the rocket can "dump" fuel is not considered. If the target is not initially incoming, then another method must be used to get the first estimate of T .

8. Interception with minimum fuel consumption.

The problem of intercepting a ballistic target with minimum fuel consumption is discussed here for an incoming target. As in section 7 the equations of motion used are those for constant gravity.

For interception, the two equations must be satisfied

$$(28) \begin{cases} \dot{X}_0 + \dot{X}_0 T = \int_0^{t_1} (T-t)a \cos p \, dt \\ \dot{Y}_0 + \dot{Y}_0 T = \int_0^{t_1} (T-t)a \sin p \, dt \end{cases}$$

as in the preceding section. There are three parameters p, t_1, T and only two equations so there is one degree of freedom in the choice of the trajectory. The third equation is obtained from the condition $dt_1/dT = 0$, as follows. Consider two neighboring interception trajectories, each satisfying (28). On the second the values p, t_1, T are replaced by $p+\delta p, t_1+\delta t_1, T+\delta T$. Since both trajectories satisfy equation (28),

$$(29) \begin{cases} (\dot{X}_0 - \int_0^{t_1} a \cos p \, dt) \delta T - (T-t_1)a(t_1) \cos p \, \delta t_1 \\ \quad + \int_0^{t_1} (T-t)a \sin p \, dt \, \delta p = 0 \\ (\dot{Y}_0 - \int_0^{t_1} a \sin p \, dt) \delta T - (T-t_1)a(t_1) \sin p \, \delta t_1 - \\ \quad - \int_0^{t_1} (T-t)a \cos p \, dt \, \delta p = 0 \end{cases}$$

When δp is eliminated, the equation results

$$(30) \left[(\dot{X}_0 - \int_0^{t_1} a \cos p \, dt) \cos p + (\dot{Y}_0 - \int_0^{t_1} a \sin p \, dt) \sin p \right] \delta T \\ (T-t_1)a(t_1)\delta t_1 = 0.$$

The case where $t_1 = T$ is an extreme case and will be excluded from consideration here. For a minimum value of t_1 ,

$$(31) (\dot{X}_0 - \int_0^{t_1} a \cos p \, dt) \cos p + (\dot{Y}_0 - \int_0^{t_1} a \sin p \, dt) \sin p = 0.$$

In formal calculus of variations, this is a transversal condition. Since the terms in parentheses are the components of the relative velocity vector for $t > t_1$, this condition may be interpreted as the condition that the relative velocity be perpendicular to the direction of thrust, for $t > t_1$. If the target follows a known maneuver, this condition may hold only at time T .

Computational routine. Choose p so that the direction of thrust is perpendicular to the initial line of sight. As before, set

$$(25) \quad E = X_0 + \dot{X}_0 t - \int_0^{t_1} (t-\tau)a \cos p \, d\tau \\ F = Y_0 + \dot{Y}_0 t - \int_0^{t_1} (t-\tau)a \sin p \, d\tau,$$

and guess t_1 . T may be either determined from p , or because the routine was set up from the preceding problem, we computed $E^2 + F^2$ until it began to increase to get the first estimate of T .

From equation (30), a new value for p was obtained, and hence δp . Then from the equations

$$(3) \left\{ \begin{aligned} \delta E &= (\dot{X}_0 - \int_0^{t_1} a \cos p \, dt) \delta T - (T-t_1)a(t_1) \cos p \, \delta t_1 \\ &\quad + \int_0^{t_1} (T-t)a \sin p \, dt \, \delta p \end{aligned} \right.$$

$$(32) \quad \left\{ \begin{aligned} \delta F &= (\dot{Y}_0 - \int_0^{t_1} a \sin p \, dt) \delta T - (T - t_1) a(t_1) \sin p \, \delta t_1 \\ &\quad - \int_0^{t_1} (T - t) a \cos p \, dt \, \delta p, \end{aligned} \right.$$

for the variations and from

$$(27) \quad \begin{cases} \delta E = -E \\ \delta F = -F, \end{cases}$$

$\delta t_1, \delta T$ are determined. The value of T is determined from (27) for subsequent iterations, rather than from the condition that $E^2 + F^2$ is a minimum. The value for p is determined each time from (31) and then δp , this value being used in (32) to reduce it to two equations in two unknowns.

9. Comments, other problems.

The above method of solution applies generally to problems wherein the velocity is not involved at the terminal point. There is also a restriction that T be not too large if the linearized gravitational acceleration is used. In these cases, R is always a decreasing function of t , vanishing for $t = T$, and the acceleration is then always to be applied initially.

There is a growing awareness that problems in calculus of variations require the maximization of an integral, and that this in turn requires the maximization of an integrand. The usual treatment of isoperimetric problems (see Margenau and Murphy [1], p. 204) suggests this. In formal calculus of variations this is done by an ϵ - δ procedure, but, particularly in the case of linear differential equations it can often be done more easily directly. The concept of maximizing the integral has received considerable impetus recently due particularly to papers by Pontrjagin [2],[3]. The author first became aware of it from a paper by Emerson [4], and the paper of

Breakwell[5] brings out this property of extremals. The integral in (18) is the functional of Pontrjagin, and the determination of c, t_1, T effects the construction of the functional.

The philosophy of solution here is that one may obtain a solution by simple considerations of this maximizing principle, and then verify that it is the solution. It should be observed that the proof given actually only deals with the nature of the extremals; the interrelations between the extremals and the manifolds whereon the terminal point lies are more complicated, and we have only discussed conditions for a stationary value.

For those interested in classic calculus of variations, the condition $1 - \cos(p^* - P) \geq 0$ is the Weierstrass condition for the steering. The strict inequality is of course never satisfied for all values of P , since $\cos(p^* - P)$ is periodic, but for practical purposes it is always satisfied since values of p which differ by 2π are equivalent.

Comments on computation. The integrals were expressed as differential equations and the system was integrated by a Runge-Kutta routine. It was selected for convenience in programming, with no attempt to attain speed. Each iteration, to obtain an approximating trajectory, requires about one second on the CDC 1604. The squared error $E^2 + F^2$ tended to diminish by a factor of 50-60 each time. The routine was assumed to have converged when $E^2 + F^2$ became less than $10^4(\text{ft}^2)$ in the examples. A further criterion should be included corresponding to equation (31) but in the examples run, which could be checked, it was not found to be necessary, probably due to the fact that in this problem, the values for the angle seem to "converge rapidly".

It usually took three to six seconds to determine a trajectory. Some trajectories were continued indefinitely to obtain an estimate of the accuracy of the routine. In the examples run, the range was a few hundred thousand feet, the time of flight about forty seconds and the initial acceleration of the rocket due to thrust about 6 g. The terminal squared error $E^2 + F^2$ reduced directly to about $10^{-10}(\text{ft}^2)$.

An interesting convergence problem was encountered in the problem of interception in minimum time. The routine to determine the trajectory was only for an extreme value of T . It seemed obvious that for some values of the first estimates for c or $\tan p$, a minimum value of T would be obtained, and for other first estimates, a maximum value of T . Consequently a series of trajectories were run, with various first estimates for p , namely $0, \pi/6, \pi/3, \pi/2, 2\pi/3, 5\pi/6, \pi$. The value of p associated with minimum T was about 1.6, and with maximum T , 3.2 (radians). For the initial estimates $p = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$, the routine converged directly to the value $p = 1.6$. For the initial value $p = \pi$, the routine converged directly to $p = 3.2$. For the initial estimate $p = 5\pi/6$, the routine did not appear to converge initially, but the successive estimates of p varied in a random fashion. Finally p came close to 14.17 and converged to this value, which is $1.6 + 4\pi$. The method of solution is essentially a Newton's method, and this is a typical behavior when the starting values are not near enough to the desired root.

These problems are among the simplest in the calculus of variations since the equations are linear in the dependent variables. The use of λ rather than the mass simplifies the differential equations further. It can be shown in several problems that if the time when the fuel is to be used is not specified, then it is to be used when the magnitude of a vector, R in this case, exceeds a certain value (this value to be determined), and the fuel is to be used at the minimum rate whenever R is below the value. It is not known whether this relation is general or not; no counterexamples are known to the authors.

The procedures for solution, using the adjoint system, can be applied to non-linear problems equally well, though no simple complete proofs exist corresponding to the ones given here. In this case the adjoint system is adjoint to the system of variational equations for the dependent variables, rather than the original system. A report is forthcoming on this; see also the paper by Breakwell [5], though he does not treat the case where

a is linear in the fuel consumption rate. The steering equation is probably first due to Lawden in published reports (see [6] for references). The method of calculating differentials is essentially that which Bliss formulated in Ballistics (see [7], Chapter V, for introductory theory, and p. 125, for early pertinent references). The theory is from lectures given at Boeing in July 1960 on the applications of the adjoint system in control. An appendix is planned, giving the computational details for this problem.

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Erratum

The upper limit of the last integral in equation (19) should be t_1 , not T .

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